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β-PLANE MAGNETOHYDRODYNAMIC TURBULENCE IN THE SOLAR TACHOCLINE

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ABSTRACT

This Letter discusses the role of a weak toroidal magnetic field in modifying the turbulent transport properties of stably stratified rotating turbulence in the tachocline. A local two-dimensional β-plane model is investigated numerically. In the absence of magnetic fields, nonlinear interactions of Rossby waves lead to the formation of strong mean zonal flows. However, the addition of even a very weak toroidal field suppresses the generation of mean flows. We argue that this has serious implications for angular momentum transport in the lower tachocline.

Subject headings: MHD — Sun: interior — Sun: magnetic fields — turbulence

1. INTRODUCTION

The solar tachocline is now considered to be of fundamental importance in driving solar magnetic activity. This thin layer of strong radial shear at the base of the solar convection zone is believed to play a key part in the generation of large-scale magnetic fields by the solar dynamo and may also play an important role in mixing processes in the solar interior. Since its discovery, there has been some progress in understanding the fundamental processes that determine the dynamics of the tachocline, although there remains much that is not understood (see, e.g., Tobias 2005; Hughes et al. 2007).

It is now generally accepted that the tachocline comprises two layers. The upper tachocline is coupled strongly to the convection zone above, is believed to have a strong toroidal magnetic field generated by the solar dynamo, and is neutrally or weakly stably stratified. The lower tachocline is very stably stratified, and less is known about the form and strength of any magnetic field. There are many open questions concerning the dynamics of both regions. These include, for the upper tachocline, the role of overshooting convection, the stability of strong toroidal fields to joint and magnetic buoyancy instabilities, and the role of the dynamo-generated magnetic field (Forgács-Dajka & Petrovay 2001); and, for the lower tachocline, the role of slow meridional flows, magnetic field, and turbulence in redistributing magnetic fields on a long timescale.

As highlighted by Spiegel & Zahn (1992), the fundamental question concerning the tachocline is that of its very existence. They point out that slow radiation-driven meridional flows would redistribute angular momentum and would, in the simplest case, lead to an appreciable inward spreading of the tachocline over the age of the Sun. They argue, however, that strong stratification would noticeably modify any turbulence, rendering it essentially two-dimensional, resulting in an anisotropic eddy viscosity (or friction), which, in turn, would prevent the spreading of the tachocline. A central question, therefore, is whether stably stratified turbulence does indeed act as a viscosity (friction) in redistributing angular momentum on long timescales. Gough & McIntyre (1998) argue instead that such turbulence would act as an *anti*friction, homogenizing potential vorticity and driving mean flows. At the heart of this argument, therefore, is the role of the fast turbulent processes in redistributing angular momentum on a long timescale.

Although this problem is of central importance for tachocline dynamics, it has been relatively uninvestigated in this context. An excellent review of the literature and explanation of the underlying physics is provided by Miesch (2005), and we give only a brief summary here. The lower tachocline can be modeled as an incompressible strongly stratified rotating fluid; flows then take the form of pancake-like structures in decoupled two-dimensional layers (e.g., Herring & Métais 1989). Two-dimensional systems are well known to form large-scale flows via the process of vortex merging (an inverse cascade)—a process that can be halted by the presence of rotation. In a rotating system, these vortex patches can be thought of as nonlinear Rossby waves that propagate zonally and disperse before merging, leading to the formation of zonal flows on large scales (e.g., Rhines 1975; Diamond et al. 2005). Some aspects of this hydrodynamic behavior persist in fully three-dimensional simulations of stably stratified turbulence in spherical shells (see Miesch 2001, 2003).

In this Letter, we examine the influence of a magnetic field on the dynamics of forced, rotating, two-dimensional turbulence. This is a potentially important issue since it is well known that even very weak magnetic fields may play an important role in modifying turbulent transport properties. As a simple local model of the deep tachocline, we consider two-dimensional dynamics on a β-plane (in common with some of the hydrodynamic models described above). We allow for the presence of a mean field aligned with the direction of potential jet formation and determine its influence on the dynamics. We present the results of high-resolution, numerical simulations of the model in order to determine the subtle effects of the aligned magnetic field in modifying the turbulent fluxes that lead to jet formation. Complementary analytical approaches have been undertaken using either closure (Diamond et al. 2007) or quasi-linear approximations (Leprovost & Kim 2007).

2. MODEL AND EQUATIONS

2.1. *Parameter Regime for the Base of the Tachocline*

As noted above, strong stable stratification leads to the generation of predominantly two-dimensional flows. To fix ideas, we summarize the important nondimensional parameters for dynamics in the lower tachocline. Although the astrophysical parameter regime is not accessible by numerical simulations, we are able to examine turbulent flows in which the correct ordering of the various physical effects is maintained. The degree of stratification in a shear flow is often measured by the Richardson number Ri; in the lower tachocline $\mathrm{Ri} \approx 10^3$, and so stratification is of central importance. The Rossby number Ro measures the ratio of inertial to Coriolis terms; in the tachocline, $\mathrm{Ro} \approx 0.1$–1 (this estimate has a degree of uncertainty

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owing to difficulties in estimating typical flow speeds). Hence, rotational effects are as important as inertial effects. The Ekman number, a measure of the relative importance of viscous to Coriolis terms, is exceptionally small ($E \approx 10^{-11}$). The magnetic Reynolds number is given by $Rm = RoE^{-1}Pm$ (where Pm is the magnetic Prandtl number). For the tachocline, $Rm \approx 10^5$–10^6 and is therefore large (even allowing for uncertainties in the size of Ro), while the fluids Reynolds number $Re = RoE^{-1}$ is even larger.

To summarize, the appropriate regime is where the flow is two-dimensional ("shellular"), rotationally influenced although not rotationally constrained, and turbulent. There are, though, no reliable estimates of the strength of the large-scale magnetic field in the tachocline, so it is of interest to determine how the dynamics may change with large-scale field strength.

2.2. *Equations and Geometry*

The governing equations of our model are the forced momentum equation for an incompressible fluid with unit density

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\nabla p + \boldsymbol{j} \times \boldsymbol{B} + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{F}, \quad (1)$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad (2)$$

together with the magnetic induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}, \quad (3)$$

where $\boldsymbol{u}$ is the velocity, $\boldsymbol{F}$ is the forcing, $\boldsymbol{\Omega}$ is the rotation, $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{j}$ is the current, and ν and η are nondimensional measures of the viscosity and magnetic diffusivity. Because the system is forced, the natural size of the velocity (and hence the significance of the sizes of ν, η, and the imposed magnetic field) does not emerge until the system has equilibrated. Nondimensional parameters involving velocities can only be calculated a posteriori.

We consider a local domain located in the lower tachocline, with coordinate axes (x, y, z) as summarized in Figure 1. Here x represents longitude (azimuth), y colatitude, and z depth. We consider a two-dimensional model with all variables independent of z. In order to include the effects of rotation in such a model, it is necessary to employ the β-plane approximation, where only the vertical component of rotation is retained and $2\boldsymbol{\Omega} = (0, 0, f + \beta y)$ (see, e.g., Pedlosky 1992). We consider a two-dimensional incompressible velocity field $\boldsymbol{u} = (\psi_y, -\psi_x, 0)$, where $\psi(x, y, t)$ is the stream function. The magnetic field has a mean azimuthal component (B_0), and so we set $\boldsymbol{B} = (B_0 + A_y, -A_x, 0)$, where $A(x, y, t)$ is the potential. The vorticity $\boldsymbol{\omega} = [0, 0, \omega(x, y, t)]$ with $\omega = -\nabla^2\psi$. Progress is made by considering the z-component of the vorticity equation together with the equation for A. These give

$$\omega_t = J(\psi, \omega) + \beta\psi_x + J(A, \nabla^2 A) - B_0\nabla^2 A_x + \nu\nabla^2\omega + G_0, \quad (4)$$

$$A_t = J(\psi, A) + B_0\psi_x + \eta\nabla^2 A, \quad (5)$$

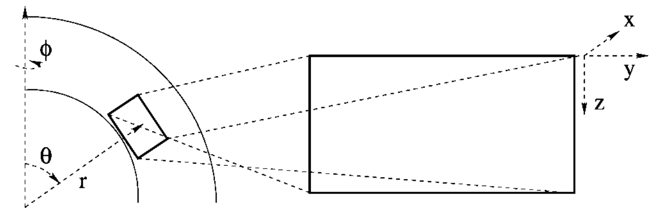

FIG. 1.—Geometry and computational domain for the local Cartesian model.

where $J(a, b) = a_x b_y - a_y b_x$ and $G_0(x, y, t)$ is the z-component of the couple.

As the model is local, we employ periodic boundary conditions. Equations (4) and (5) are solved using standard pseudospectral techniques on a parallel computer. This enables us to access turbulent parameter regimes. The forcing G_0 is designed to sample spectral modes with wavenumbers $14 \leq k_x$, $k_y \leq 20$ and amplitude G_1. For two reasons, and unlike some other simulations of purely hydrodynamic β-plane turbulence, we do not include any sink term to remove energy at large scales. First, for the tachocline it is expected that (purely hydrodynamically) any large-scale zonal flow would dominate the small-scale turbulence, and so it makes no sense to suppress the strength of this flow arbitrarily. Second, as we are interested in determining the strength of magnetic field that can inhibit the generation of mean flows, we err on the side of caution and ignore all arbitrary suppression mechanisms.

3. RESULTS

The dynamics of the model system is controlled by the parameters G_1, β, and ν for purely hydrodynamic simulations and, in addition, by B_0 and η for MHD simulations. In this Letter, we hold constant the forcing $G_1 = 2.0$ and the viscosity $\nu = 10^{-4}$. The emerging flows are therefore turbulent and require a high resolution (1024^2) for accurate computation. In § 3.1, we examine the well-understood hydrodynamic case for two choices of β, before examining the effects of adding a magnetic field in § 3.2.

3.1. *Purely Hydrodynamic Solutions*

The purely hydrodynamic ($B_0 = A = 0$) system is started from rest ($\omega = 0$) with a prescribed forcing that drives the system away from the trivial state. As shown in many previous studies, the dynamics follows an inverse cascade from small to large scales, with many vortex-merging events leading to the formation of zonal flows (e.g., Vallis & Maltrud 1993). This merging process can be characterized by Rossby wave interactions and decorrelations, where the small-scale Reynolds stresses correlate and act as "antifriction" to drive a large-scale flow. The zonal flow saturates on a (long) viscous timescale, and the system reaches a statistically steady state, shown in Figure 2 for $\beta = 5$ and $\beta = 50$. It is clear that a large-scale zonal flow has emerged in both cases, with a latitudinal (y) extent that depends on the value of β and is consistent with the Rhines length $L \sim (U/\beta)^{1/2}$, where U is a typical velocity scale. We also note here that for both cases most (>97%) of the kinetic energy is contained in the mean (zonal) flows and so the system evolves to a shear-dominated state. The aim of this Letter is to investigate whether this behavior persists when magnetic fields are present and, if not, to determine the field strength at which this picture breaks down.

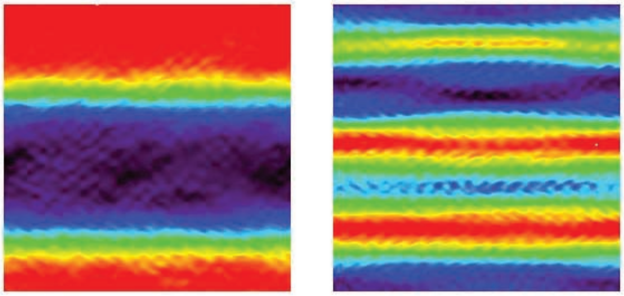

FIG. 2.—Color-coded density plots of $u = \psi_y$ for $\beta = 5$ (*left*) and $\beta = 50$ (*right*). Red indicates strong prograde flows, while blue indicates strong retrograde flows. Because $u \gg v = -\psi_x$, these plots are representative of the flow field.

3.2. *The Role of a Weak Toroidal Field*

In this section, the dynamics of the MHD ($B_0 \neq 0$) system is investigated (with $\beta = 5$) and compared with that of the hydrodynamic system described in § 3.1. Again the system is evolved from rest and allowed to reach a statistically steady state. This experiment is repeated for a range of values of the imposed toroidal field, $10^{-4} \leq B_0 \leq 10^{-1}$. The final state of the system is shown in Figure 3 for four choices of the imposed field, for $\eta = 10^{-4}$. For $B_0 = 10^{-3}$, the inverse cascade proceeds in a similar manner to the hydrodynamic case and the zonal flow emerges and saturates after a viscous time; the final state is essentially identical to that for $B_0 = 0$. However, the top two panels demonstrate that once B_0 exceeds some threshold value, the dynamics is very different. For $B_0 = 10^{-2}$ and 10^{-1}, the inverse cascade is halted by the presence of the magnetic field and the solution rapidly settles down (in an advective time) to a state of two-dimensional, small-scale MHD turbulence, characterized by the nonlinear interaction of driven Rossby and Alfvén waves. Here the dynamics is irregular with large fluctuations about a small mean. The kinetic energy is much smaller than for the cases with inverse cascades, and the energy is contained in small scales.

Figure 4 shows the mean flow $\bar{u}(y)$ for six values of the imposed field. For small values of B_0, the mean flow persists, with largely the same length scale and amplitude, while for B_0 larger than some threshold value (B_*, say) the mean flow is almost completely suppressed. Here the field acts to remove the correlations that lead to the generation of mean flows. This mechanism is investigated in detail in a subsequent paper and so is only briefly summarized here. The magnetic field is wrapped up and amplified into small-scale fields by the driven turbulence. This small-scale field exerts a Maxwell stress on the fluid that opposes the sense of the correlations that drive the mean flow. If this Maxwell stress is large enough, then the Reynolds stresses can be canceled (on average) and the mechanism for driving the mean zonal flows switched off. An alternative description (see Diamond et al. 2007) is that the field introduces Alfvén waves that compete with the dispersive Rossby waves generating the mean flows. The relative importance of the Alfvén and Rossby waves depends not only on the magnetic field strength but also on the length scales on which the interactions are occurring. We stress that this is a subtle nonlinear interaction mechanism—because the magnetic field is aligned with the emerging flows, it does not act directly to oppose the jets.

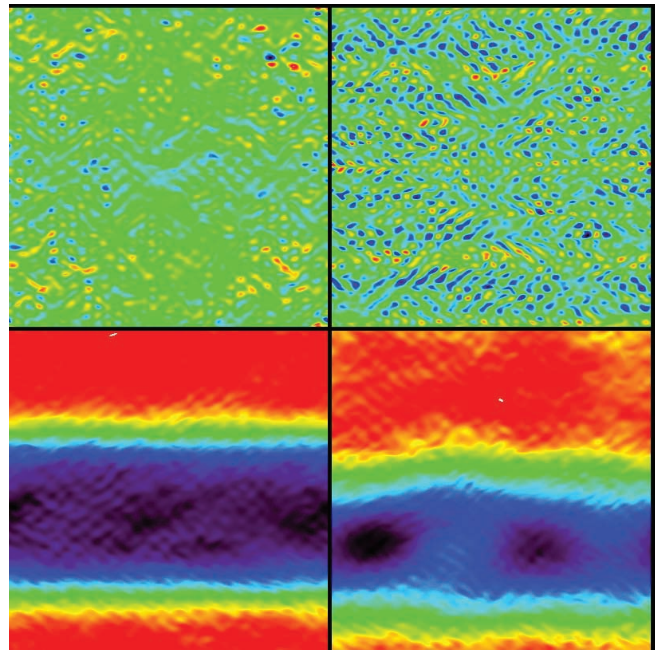

FIG. 3.—Same as Fig. 2, but with an added toroidal field B_0: $B_0 = 0$ (*bottom left*), $B_0 = 10^{-3}$ (*bottom right*), $B_0 = 10^{-2}$ (*top left*), and $B_0 = 10^{-1}$ (*top right*). Here $\eta = 10^{-4}$.

Given the uncertainty of the strength of the magnetic field in the deep tachocline, it is important to determine the dependence of the threshold B_* on parameters. In particular, it is known that for a number of turbulent MHD systems (e.g., Cattaneo & Vainshtein 1991) the molecular diffusion (often characterized by Rm) plays an important role in determining a threshold field strength. Note, however, that for the system we are considering, Rm can only be calculated a posteriori and, furthermore, it depends crucially on whether the flow inverse cascades or not. It is therefore more natural to determine a scaling with the input parameters—here we choose to vary η and keep all other parameters fixed.

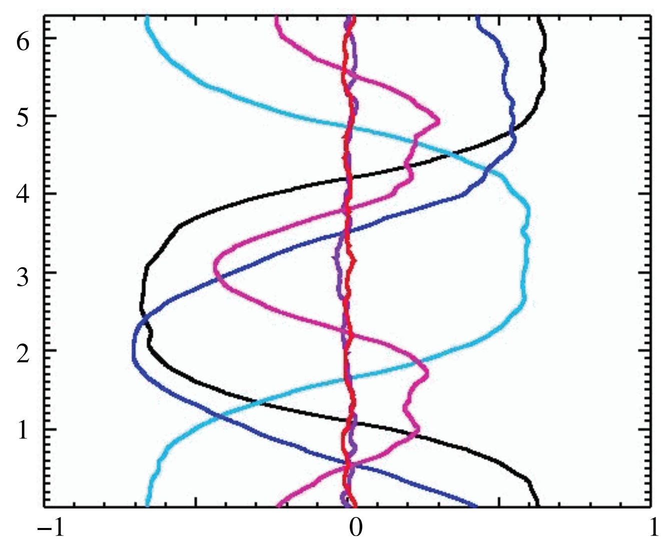

FIG. 4.—Suppression of mean flows. Mean flow $\bar{u}(y)$ for $B_0 = 0$ (*black*), 10^{-4} (*dark blue*), 10^{-3} (*light blue*), 5×10^{-3} (*pink*), 10^{-2} (*red*), and 10^{-1} (*purple*). Here $\eta = 10^{-4}$.

Figure 5 shows the result of a series of simulations with a range of B_0 and η, with the final states being characterized by the presence or absence of a large-scale mean flow. There is a well-defined scaling, namely, $B_*^2 \propto \eta$. The turbulence dynamics exhibits two characteristic behaviors, namely, those of two-dimensional MHD and two-dimensional Rossby wave turbulence. The transition between the two occurs when $k\tilde{v}_A \approx \beta k_x/k^2$, where k is the wavenumber and $\tilde{v}_A$ the Alfvén velocity of the fluctuating field (Diamond et al. 2007). This balance sets the turbulent Alfvén frequency to be comparable with the Rossby wave frequency. This occurs at the length scale $l_{RM}^2 \approx (\tilde{v}_A/\beta)$, where l_{RM} is the magnetic Rhines scale. Further assuming that the fluctuating field energy is a factor of Rm larger than that in the mean, as is typical for weakly magnetized two-dimensional MHD turbulence, we find $l_{RM}^2 \approx \mathrm{Rm}^{1/2} v_A/\beta$. A posteriori calculations of Rm, based on the amplitude of the turbulent fluctuating flows, demonstrate that the theoretical scaling is indeed consistent with the numerical results. This result has implications for the dynamics and transport properties of stably stratified, rotating MHD turbulence. The particular implications for the tachocline are discussed below.

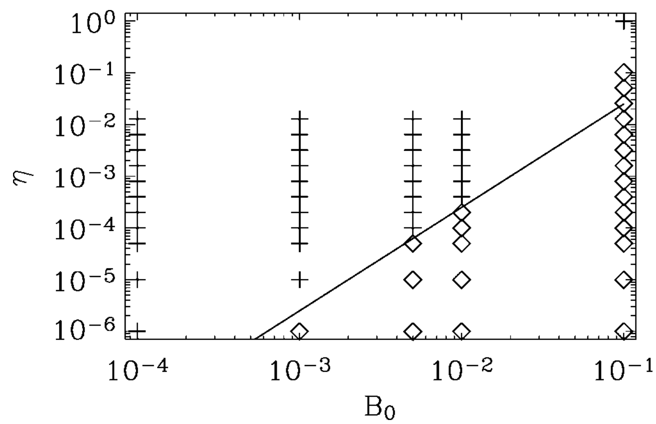

FIG. 5.—Scaling law for the transition between forward cascades (*diamonds*) and inverse cascades (*plus signs*). The line is given by B_0^2/η = constant.

4. DISCUSSION

By considering a local model of rotating, stably stratified MHD turbulence, we have demonstrated that a weak azimuthal magnetic field suppresses the formation of zonal flows. Thus, in the MHD state, turbulence no longer acts as an antifriction. The combination of two-dimensionality and large Reynolds numbers leads to the generation of a strong small-scale field. This in turn generates Maxwell stresses that, on average, cancel the Reynolds stresses. This almost exact cancellation means that the turbulence plays no role in the transport of momentum—and therefore acts neither as an antifriction or a friction. Because the ratio of the energies in the small- to large-scale field is controlled by the small magnetic diffusivity, even a very weak large-scale field can suppress the hydrodynamic mean flows.

These results have significant implications for the tachocline, the dynamics of which involves processes on timescales ranging from days to millions of years. In the upper tachocline, all of the processes are "fast," by which we mean that they take place on a timescale of, at most, a few hundred years; the lower tachocline, by contrast, involves both fast and slow dynamics. Of necessity, any modeling of the slow dynamics must invoke parameterization of the fast processes. Our study has explicitly modeled the net effect of one of the fast processes occurring in the lower tachocline. We have shown that the role of magnetized turbulence cannot be simply parameterized either by friction (anisotropic or otherwise) or antifriction. Indeed, there appears to be no net turbulent transport of angular momentum. Thus, we conclude that confinement of the tachocline is more likely to occur by slow processes. The most likely alternative is that a fossil magnetic field in the solar interior acts to inhibit the spreading of the tachocline (Rüdiger & Kitchatinov 1997). This approach, though, also has some problems, with the shear-generated toroidal field potentially becoming unstable to $m = 1$ Tayler instabilities or leading to states for which the differential rotation propagates inward along magnetic field lines (MacGregor & Charbonneau 1999; Brun & Zahn 2006).

Of course, our model is a simplification of the solar tachocline, with the flows constrained to be strictly two-dimensional and local. In subsequent investigations, we shall relax these assumptions and study turbulent diffusion and angular momentum transport in driven shallow-water MHD systems (see, e.g., Gilman 2000) in both local and global geometries.

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